Structured Propositions and a Semantics for Unrestricted Impure Logics of Ground

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Abstract

I show that the assumption of highly structured propositions can be leveraged to provide a unified semantics for various propositional logics of impure ground in a very expressive and flexible way. It is shown, in particular, that the induced models are capable of capturing an infinitude of grounding facts that follow from unrestricted logics of ground, but, due to certain artificial restrictions, are left unaccounted for by the existing semantics in the literature. It is also shown that our models, unlike the ones in the literature, are easily extendable to capture certain distinct views about iterated as well as identity grounding.

1 Introduction

There is a popular view in analytic philosophy, going back to Russell (1903), according to which propositions are highly structured, somewhat reflecting the structure and identity conditions of the sentences that express them. Call the structured propositions along these lines *Russellian* (King, 2019; Kaplan, 1989 [1977]). In recent decades, many seminal works in the philosophy of language and metaphysics have assumed or argued for Russellian propositions in various contexts, ranging from attitude operators to different kinds of metaphysical priority, such as essence and ontological dependence (see, e.g., King, 1996, 2009; Soames, 1987; Fine, 1995, 1980, 1994; Kaplan, 1989 [1977]; Salmon, 1986).

Grounding, on the other hand, is a more recent notion in metaphysics, often taken to be a non-causal relation that holds between certain truths or facts and certain others, somehow reflecting a sense of 'fundamentality' or 'explanation' between them (see, e.g., Fine, 2012a; Rosen, 2010; Audi, 2012, for comprehensive introductions to the notion of ground).[1] The conception of ground which takes the relata of the grounding relation to be propositions is sometimes called *representational* or *conceptual*; the *worldly* conception concerns entities such as states

[1]While the nature of grounding and its relations to other notions such as explanation or fundamentality is an intricate issue that has been subject to extensive discussions in the literature (see, e.g., Rosen, 2010; Woods, 2018; Fine, 2012a; deRosset, 2013; Sider, 2011; Maurin, 2019, for different readings of ground based on explanation or fundamentality, their relationships with one another, and some of the complications involved in laying out those relationships), in this paper we stay fairly neutral in this regard, and appeal to either of readings mainly for illustrative purposes.

of affairs or situations (Correia, 2017, p. 508). The kind of logics that take into account the logical structure of the relata of grounding relations are often called *impure*; *pure* logics ignore such complexities (Fine, 2012a, p. 54).

There is another important set of distinctions between grounding relations that has been studied in the literature, and we briefly introduce here (see, e.g., Fine, 2012a, pp. 52-4 for a detailed discussion of these variations and their differecnes). To start with, a number of truths are said to *fully* ground a truth if the latter somehow holds completely in virtue of the former and nothing else; a truth *partially* grounds another if it does so fully, standalone or together with other truths. Another distinction is between mediate and intermediate grounds. Grounds of a truth are *immediate* if there's no mediating truth between them and what they ground; otherwise, they constitute *mediate* grounds, as if there's a 'chain' of immediate grounds involved. Finally, some truths are *strict* grounds of some others if they are, in a sense, more 'fundamental' or 'basic'; otherwise, the grounds are *weak*. Put in terms of explanation, we can think of strict ground as, in the words of Fine (2012a), strict ground is one that "takes us down in the explanatory hierarchy," whereas weak grounds "may also move us sideways in the explanatory hierarchy" (*ibid*, p. 52). Finally, the conception of ground that allows *any* proposition, regardless of its truth value, as the relata of ground is called *non-factive*; the *factive* variant only works with truths, i.e., true propositions.

In this paper, I study an intimate relationship between Russellian propositions and *impure* logics of representational ground. The main focus of the paper is on propositional logics of ground.[2] More specifically, we are concerned with *strict partial* grounding relations: non-factive immediate ($\prec$), factive immediate ($\prec_f$), non-factive mediate ($\prec_m$) and factive mediate ($\prec_{fm}$). (Hereafter we use 'ground' to indicate strict partial ground unless stated otherwise; the specific type will be mentioned as needed.) We take the notion of ground as a primitive, i.e., not reducible to any other notion.

I show that models of Russellian propositions can be used to semantically accommodate an infinitude of grounding facts that follow from unrestricted logics of impure ground, but are left unaccounted for in the existing semantics, found in Correia (2017); Krämer (2018); deRosset and Fine (2022), due to certain artificial restrictions inherited from the languages they work with. Moreover, it is shown that our models, unlike the ones in the literature, can be very easily extended to capture certain distinct philosophical views about, e.g., iterated as well as identity grounding.[3]

[2]Some of seminal the works on the quantificational logics of ground are as follows: Fine (2012a); Korbmacher (2018a,b); Fritz (2019, 2021); Goodman (2022); Litland (2022).

[3]It might strike the reader, at this point, that Russellian propositions, as favored by the author cited earlier, have now been known to lead to the so-called Russell-Myhill paradox (as shown in, e.g., Goodman, 2017; Dorr, 2016; Hodes, 2015; Russell, 1903), and thus this might cast doubt on the conceptual value of the results to be explored in this paper. This, however, shouldn't worry us because (i) the Russell-Myhill result doesn't emerge at the level of propositional logic without quantification, so it shouldn't concern us in this paper, and (ii) the named paradox can be avoided under a different background type theory which is more ground-friendly and which saves Russellian propositions. We have discussed this issue at some length at the end of the

The sensitivity of impure ground to the structure of propositions is easily detectable once the naive principles are laid down (see Fine, 2012a, Sections 1.6-1.7 for an early discussion of these principles). For instance, it is often argued that a conjunctive truth $\phi \wedge \psi$ is grounded by each of its conjuncts ϕ and ψ (temporarily call this CG_f: $\phi \prec_f (\phi \wedge \psi) \;\wedge\; \psi \prec_f (\phi \wedge \psi)$), a disjunctive truth $\phi \vee \psi$ by either of its true disjuncts ϕ or ψ (DG_f: $(\phi \prec_f \phi \vee \psi) \;\vee\; (\psi \prec_f \phi \vee \psi)$), a doubly negated truth $\neg\neg\phi$ is grounded by the doubly negated truth ϕ (NG_f: $\phi \prec_f \neg\neg\phi$), and that no proposition grounds itself (IG_f: $\phi \nprec_f \phi$). Now, from IG_f and NG_f it follows that ϕ and $\neg\neg\phi$ can't be the same truth, and from CG_f and DG_f the same follows for any pair of sentences from ϕ, $\phi \vee \phi$ and $\phi \wedge \phi$.[4]

So, we quickly get a few boundaries surrounding the issue of propositional granularity under considerations of ground. As a result, certainly, coarse-grained accounts, such as the once-popular intensionalism which identifies necessarily equivalent propositions (see, e.g., Montague, 1969), and its close, more recently popularized cousin, Booleanism, which identifies propositions that are provably logically equivalent (see, e.g., Dorr, 2016) can't be consistently adopted under the principles above.

In general, as we will see along the way, from the propositional logics of impure ground, along with minimal principles of propositional identity, it follows that propositions ought to be significantly structured—in fact, sometimes as structured as Russellian propositions (see Theorem 3.1). Moreover, recently Fritz (2021) has shown that higher-order formulations of the principles of ground, in fact, entail certain higher-order instances of Russellian propositions. It can be said that the propositional and higher-order logics of immediate ground together portray a scattered picture of propositions that is most straightforwardly and systematically captured by Russellian propositions.[5]

Here's how the paper is organized. In Section 2, I informally address certain expressive shortcomings of the existing semantics of impure logics of ground. In Section 3, I rigorously introduce the language and lay down the immediate and mediate logics of ground, both non-factive and factive variants. In the same section, I establish certain structural results derived from logics of ground, lay down some identity principles for Russellian propositions and show that the latter systematically capture the former. Section 4 concerns semantics; it introduces propositional models for Russellian propositions, uses them to provide a semantics for the unrestricted logics of ground and discusses some meta-results such as

paper.

[4]See Section 3 for more general formulations of these principles; the naming used here is temporary.

[5]Other works on the logic of ground that impose some kind of structural hierarchies on propositions are Poggiolesi (2016) and Correia (2017), though both end up with more relaxed structures on propositions in comparison to Russellian propositions. Poggiolesi (2016) appeals to the notion of 'g-complexity' for this but it's not clear if the resulting account fully appreciates the level of complexity of propositions that naturally emerges from the principles of grounding and the minimal principles of identity—i.e., the results in Theorem 3.1. It's also unclear how Pogiolessi's account will perform when it comes to the semantic issues that are at stake in this paper. We will leave these open here. We will discuss Correia (2017) in more detail later in the paper.

soundness, consistency and completeness. Section 5 proposes various desirable extensions of logics of impure ground and their semantics and addresses some systematic difficulties of the existing semantics in the literature in undergoing similar extensions. Section 6 concludes the paper. The appendices collect all the principles of grounding and propositional identity and establish some of the technical results in the paper.

2 Present Semantics and their Shortcomings

While the semantics of pure logics of ground has been well studied and somewhat settled (see, e.g., Fine, 2012b), impure logics, and in particular, their representational variants, remain fairly underexplored, with only a few recent attempts on offer to semantically account for them (Correia, 2017; Krämer, 2018; deRosset and Fine, 2022). But, even though these works mark considerable progress in the study of the impure logics of ground, all these semantic accounts suffer from certain expressiveness limits, complying with the restricted languages or logics that they're supposed to capture. To see this, we should first see what limits are imposed on the languages and logics that these semantics attempt to capture.

In general, there is a tendency in the literature on the impure logics of ground to substantially impoverish the languages in which the principles are expressed, mainly allowing for statements of grounding in which the relata of ground contain truth-functional connectives, but not connectives such as grounding itself or propositional identity (see, e.g., Schnieder, 2011; Krämer, 2018; Correia, 2017; Poggiolesi, 2020; Lovett, 2020; deRosset and Fine, 2022). As a result, an infinitude of grounding truths which live beyond these artificial restrictions are dismissed by these logics.

To illustrate this, suppose ϕ and ϕ are two sentences. Then, clearly, $\phi \prec_f (\phi \wedge \psi)$ follows from CG_f. But, by the same count, we would also expect the proposition expressed by ϕ to ground the one expressed by $\phi \wedge (\phi \prec_f (\phi \wedge \psi))$—i.e., $\phi \prec_f (\phi \wedge (\phi \prec_f (\phi \wedge \psi)))$; after all, ϕ *is* a conjunct of $\phi \wedge (\phi \prec_f (\phi \wedge \psi))$. In a similar fashion, we can consider consequences of CG_f, DG_f or IG_f where the relata of the grounding symbol are statements containing sentential identity $\approx$. For instance, by CG_f, the proposition expressed by $\phi \approx \phi$ grounds the one expressed by $(\phi \approx \phi) \wedge \psi$, and by DG_f, the latter grounds the proposition expressed by $(\phi \vee \phi) \vee ((\phi \approx \phi) \wedge \psi)$—thus: $(\phi \approx \phi) \prec_f ((\phi \approx \phi) \wedge \psi)$ and $\big((\phi \approx \phi) \wedge \psi\big) \prec_f \big((\phi \vee \phi) \vee ((\phi \approx \phi) \wedge \psi)\big)$. Finally, by IG_f, none of the propositions expressed by these grounding statements grounds itself. Clearly, an infinitude of examples such as these and even more complex ones can be given.

Now, as natural and plausible as these are, the current model theories (found in Krämer, 2018; Correia, 2017; deRosset and Fine, 2022) cannot capture them. The main reason for this is that they, much like the other works cited above, work with logics in which such grounding statements are not grammatically well-formed, so the principles are consequently restricted as well. But why impose such draconian, artificial restrictions on language or logic? As mentioned by some of the authors, nothing other than convenience in treatment seems to play role in such restrictions. In fact, there is no trace of such restrictions in some of the pioneering works that put forward and argue for these principles (e.g., Fine, 2012a).

Finally, some philosophers have put forward certain distinct views about iterated grounding and the grounds of identity statements. Consider, for instance, the view endorsed by, e.g., Bennett (2011), according to which a grounding truth like $\phi \prec_f \psi$ is grounded by its ground ϕ; that is: $\phi \prec_f (\phi \prec_f \psi)$. Or consider the view due to Wilhelm (2020a), according to which identity statements of the general form 'a is a' are ('entity-')grounded by a, where a can be any entity, such as an individual, fact, proposition, or relation. One might pick up this idea and apply it to the context of fact-grounding (e.g., by arguing that fact-grounding is a special form of entity-grounding, where the entities are limited to facts or propositions), to come up with a similar principle according to which the truth expressed by the (propositional) identity $\phi \approx \phi$ is grounded by the one expressed by ϕ; that is, $\phi \prec_f (\phi \approx \phi)$.[6]

Again, the existing models all fall short of capturing such views simply because their languages don't even allow for forming them.

In Section 5.3 we discuss these restrictions in the existing semantics and the prospects of lifting them in more detail.

3 Language and Logics

In this section, I lay down the language and logics of different variants of ground in a rigorous way and establish some structural results derived from the logics of ground. I also lay down some principles characterizing Russellian propositions, and show that they entail all the structural results derived from the logics of ground.

We assume that we have infinitely many sentential variables $p_1, p_2, ...$, and show the set that contains them all with AT. Here's a presentation of our language $\mathcal{L}$:

Definition 3.1 (Language $\mathcal{L}$)**.** The formulas of $\mathcal{L}$ are constructed as follows:

1. p_i is a formula, where $i \in \mathbb{N}$,
2. If ϕ and ψ are formulas, then so are $\neg\phi$ and $\phi \circ \psi$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$.

Aside from the familiar Boolean cases, formulas of the form $\phi \approx \psi$, $\phi \prec \psi$ and $\phi \prec_m \psi$ respectively represent statements of *propositional identity*, *immediate* and *mediate* grounding.

[6] Note that Wilhelm (2020a) argues for the adoption of entity-grounding, where all kinds of entities can enter into grounding relations, *over* the more familiar notion of fact-grounding that is at stake in this paper, not their coexistence. Moreover, one might argue that fact-grounding isn't a form of entity-grounding (see, e.g., Kiani, MSa, for an argument on this). Regardless of these, all that matters to us here is the possibility of verifying or falsifying such principles at the level of semantics—something that the existing semantics in the literature seem to fail to do.

Notice that our connectives are all given as *primitive* symbols of the language; thus, e.g., we don't have $\phi \to \psi$ as a shorthand for $\neg\phi \vee \psi$. (Of course, as expected, from our logic it will follow that these are truth-functionally equivalent.) We will return to the importance of this choice at the end of Section 5.1.

We've mentioned since the beginning of the paper that our models are going to treat propositional identity along the lines of Russellian propositions, according to which propositions exhibit the same structure and identity conditions that their underlying sentences do. Our background logic of propositional identity, accordingly, would be expected to capture Russellian propositions, hence, e.g., considering all non-identities of the form $\phi \not\approx \neg\neg\phi$, $\phi \not\approx \phi \vee \phi$, $\neg\phi \not\approx (\psi \wedge \gamma)$ and $\neg\phi \not\approx (\psi < \gamma)$, all reflecting similar corresponding syntactic non-identities, as theorems.

We will eventually do so (Section 3.2), but for now, it's worth seeing that even under certain plausible, minimal principles of identity, in general, the logic of immediate ground formulated above entails a considerable amount of propositional structure, at times even conforming to Russellian propositions. When put together with the higher-order parallel result due to Fritz (2021), this portrays a picture of structured propositions implied by considerations of ground, which is most straightforwardly and systematically captured by Russellian propositions; a result that we will establish shortly.

The principles of propositional identity that we endorse are schematically stated as follows:

Minimal Principles of Propositional Identity (MPPI)

1. $\phi \approx \phi$ — Ref
2. $(\phi \approx \psi) \to (\psi \approx \phi)$ — Sym
3. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \to (\phi \approx \gamma)$ — Tr
4. $((\phi \approx \psi) \wedge \phi) \to \psi$ — IdTr
5. $(\phi \approx \psi) \to (\neg\phi \approx \neg\psi)$ — IdSt$_1$
6. $((\phi \approx \psi) \wedge (\gamma \approx \theta)) \to ((\phi \circ \gamma) \approx (\psi \circ \theta))$, where $\circ \in \{\wedge, \vee, \to, \leftrightarrow, <, <_m, \approx\}$ — IdSt$_2$

The principles will be given in the assumption of all theorems of classical propositional calculus in the background, which we cite as PC.[7] The first three principles are the standard principles of identity—reflexivity, symmetry and transitivity. IdTr says if two propositions are identical the truth of one implies the

[7] Strictly, PC extends the theorems of the usual classical propositional calculus with Boolean connectives by allowing to express identity as well as grounding statements. So, for example, In PC, from $\phi \approx \psi$ and $(\phi \approx \psi) \to \gamma$ follows γ, using *Modus Ponens*. See Fritz (2021); Dorr et al. (2021) as examples of works that use extended versions of propositional calculus, similarly or even more generally than here, in formulating logics in languages with higher expressive power.

truth of the other, and the last two are schemata take our connectives to be functional in behavior: for example, if ϕ and ψ are the same propositions, their negations are the same as well.

3.1 Non-Factive Ground

We now introduce the notion of immediate grounding and its unrestricted non-factive logic. After that, we state some structural results that follow from the logic and the principles of identity stated above.

Immediate grounding concerns the relation of grounding that is intimate and holds between two propositions without any other propositions mediating this; *mediate* ground allows for such mediation and can be defined in terms of 'chains' of immediate grounding statements (see Fine, 2012a, pp. 50-1, for a discussion of mediate and immediate grounding). To illustrate, all the principles informally sketched in Section 2 (IG, CG, DG and NG) exhibit principles of immediate, as well as mediate, grounding. On the other hand, since, e.g., $\phi \prec (\phi \wedge \psi)$ and $(\phi \wedge \psi) \prec ((\phi \wedge \psi) \wedge \gamma)$ are both instances of immediate grounding obtained using CG, by forming a 'chain' one can deduce $\phi \prec_m ((\phi \wedge \psi) \wedge \gamma)$; though, in this case, a parallel immediate grounding relation doesn't hold. As expected, mediate but not immediate grounding is transitive.

Earlier we mentioned the distinction made between factive and non-factive grounding. As is expected, *factive* grounding concerns only true propositions, i.e., truths, or facts, whereas *non-factive* grounding allows for the relata of ground to be false. While factive grounding is what the literature is often interested in, non-factive grounding represents a more fundamental notion in terms of which factive grounding can be defined, but not necessarily *vice versa* (see, e.g., Fine, 2012a, pp. 48-50, for an introduction to this distinction and a discussion their interdefinability).

Here's the unrestricted logic of non-factive grounding (see, e.g., Wilhelm, 2020b; Fritz, 2021; Correia, 2017, for factive variants of these):[8][9]

[8] It should be noted that not all of these works introduce new principles; for example, Wilhelm (2020b) only works with some of these principles to derive certain inconsistencies against a particular coarse-grained view of propositional identity. Nevertheless, these are some of the works that embrace such principles in their analyses.

[9] One might take issue with CG with an instance such as the following: $\phi \prec (\phi \wedge \neg\phi)$. It might be thought that even in non-factive grounding where we don't necessarily deal with facts, A grounds B if A *would've* grounded B, were they true, or that there is a possible world where A is true and explains B, etc. While such readings seem intuitive at first glance, it's not clear if they can be developed consistently, or if we should push for a factive interpretation of non-factive grounding in the first place. In fact, Fine (2012a, p. 49) attempts to reduce non-factive to factive ground in a similar way to the ones above and he runs into difficulties, which essentially leads him to leave the notion of non-factive ground as a primitive notion (while he earlier defines the notions of factive in terms of non-factive ground, somewhat like ours). In general, the literature doesn't seem to support such reductions. In fact, some explicitly argue for a primitive reading of non-factive grounding. For instance, Litland (2017) chooses non-factive over factive grounding as primitive and works towards solving the so-called 'problem of iterated ground' using the notion of 'zero-grounded'. One might respond this way: "But inconsistencies cannot be grounded; why would one want to account for such grounding relations? What would be the point?" Non-factive

Unrestricted Non-Factive Immediate Ground (UNIG)

1. $\phi \nprec \phi$ IG
2. $(\phi \prec (\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ CG
3. $(\phi \prec (\psi \vee \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ DG
4. $(\phi \prec \neg(\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ NCG
5. $(\phi \prec \neg(\psi \vee \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ NDG
6. $(\phi \prec \neg\neg\psi) \leftrightarrow (\phi \approx \psi)$ NG

Here is the unrestricted propositional logic of non-factive partial mediate ground (see, e.g., Fine (2012a); Krämer (2018); Schnieder (2011) for these):

Unrestricted Non-Factive Mediate Ground (UNMG)

1. $\phi \nprec_m \phi$ IG_m
2. $((\phi \prec_m \psi) \wedge (\psi \prec_m \theta)) \rightarrow \phi \prec_m \theta$ TRG_m
3. $(\phi \prec_m (\phi \wedge \psi)) \wedge (\psi \prec_m (\phi \wedge \psi))$ CG_m
4. $(\phi \prec_m (\phi \vee \psi)) \wedge (\psi \prec_m (\phi \vee \psi))$ DG_m
5. $(\neg\phi \prec_m \neg(\phi \wedge \psi)) \wedge (\neg\psi \prec_m \neg(\phi \wedge \psi))$ NCG_m
6. $(\neg\phi \prec_m \neg(\phi \vee \psi)) \wedge (\neg\psi \prec_m \neg(\phi \vee \psi))$ NDG_m
7. $\phi \prec_m \neg\neg\phi$ NG_m

We can now see how the immediate logic imposes a considerable amount of structure on propositions.

grounding can perhaps be explained as a type of relation between propositions, as it were, which could be explained in terms of facts whenever those propositions happen to be true. In fact, this idea doesn't seem too far-fetched; in the recent literature, many take factive grounding as a relation between *true* propositions (Correia, 2017; Fritz, 2021, 2019; Wilhelm, 2020b; Litland, 2022; Woods, 2018). It would only seem plausible to consider non-factive grounding as a relation between propositions. Indeed, Litland (2022, footnote 3) explicitly sketches a novel reading of non-factive grounding exactly for the kind of suspicious cases such as ours along these lines and in terms of "impossible grounds": "One might want to work with a yet wider notion of non-factive ground where contradictory propositions like $p\&{\sim}p$ and $q\&{\sim}q$ can be distinguished by their having different impossible grounds—$p, {\sim}p$ and $q, {\sim}q$ respectively". Finally, if someone is still unhappy with principles like CG at a conceptual level due to such cases, one can still appreciate them for their formal utility, as they can underlie the other notions of grounding, such as factive and mediate grounding, and also a provide a powerful formal semantics for the propositional logics of ground, which is our goal in this paper. Essentially, one can consider developing a logic for this relation that fairly behaves like the relation of fact-grounding but holds between propositions in order to formally underlie and capture the logics of fact-grounding—whether or not such a relation between propositions is metaphysically intelligible. Thanks to an anonymous referee for drawing my attention to this issue.

Theorem 3.1. *The followings are theorems of the non-factive logic of immediate ground plus PC and the minimal principles of identity stated above, where, in all cases* $\circ \in \{\wedge, \vee\}$*. In other words, the following can be derived from* $\mathrm{MPPI} \cup \mathrm{UNIG}$*:*

1. $\phi \not\approx \neg\neg\phi$
2. $\phi \not\approx (\phi \circ \phi)$
3. $(\neg\phi \approx \neg\psi) \rightarrow (\phi \approx \psi)$
4. $(\gamma \not\approx \psi) \rightarrow (\neg\phi \not\approx (\gamma \circ \psi))$
5. $\neg\phi \not\approx (\phi \circ \phi)$
6. $\big((\phi \not\approx \psi) \wedge ((\phi \circ \psi) \approx (\gamma \circ \theta))\big) \rightarrow \big((\phi \approx \gamma) \wedge (\psi \approx \theta)\big) \vee \big((\phi \approx \theta) \wedge (\psi \approx \gamma)\big)$
7. $((\phi \circ \phi) \approx (\gamma \circ \gamma)) \rightarrow (\phi \approx \gamma)$

Notice that these theorems all express cases of non-identities where only conjunctive and disjunctive propositions are at stake (reflected by the condition that $\circ \in \{\wedge, \vee\}$). This is due to the fact that our logic only posits principles of conjunctive and disjunctive grounds; if we had similar principles regarding grounds of other types of propositions, we could've easily extended these results to retain even more structure and get closer to Russellian propositions (we discuss extensions of this nature in Section 5).

3.2 Russellian Propositions

We now posit a set of principles that characterize Russellian propositions; we will see all the non-identities above follow from these principles.[10]

Russellian Propositions (RP)

Axioms

1. Theorems of propositional calculus — PC
2. $\phi \approx \phi$ — Ref
3. $(\phi \approx \psi) \rightarrow (\psi \approx \phi)$ — Sym
4. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \rightarrow (\phi \approx \gamma)$ — Tr
5. $((\phi \approx \psi) \wedge \phi) \rightarrow \psi$ — IdTr
6. $((\phi \circ \psi) \approx (\gamma \circ \theta)) \leftrightarrow ((\phi \approx \gamma) \wedge (\psi \approx \theta))$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, <, <_m, \approx\}$ — Str_1

[10] It well may be the case that this isn't a complete axiomatization of Russellian propositions in our limited language, but the present principles arguably capture most if not all possible cases that come to mind, and in any case, are more than enough for our purposes here.

7. $(\neg\phi \approx \neg\psi) \leftrightarrow (\phi \approx \psi)$ Str$_2$

8. $(\phi \circ_1 \psi) \not\approx (\gamma \circ_2 \theta)$, where $\circ_1 \neq \circ_2 \in \{\wedge, \vee, \to, \leftrightarrow, \prec, \prec_m, \approx\}$ Str$_3$

9. $\neg\phi \not\approx (\psi \circ \gamma)$, where $\circ \in \{\wedge, \vee, \to, \leftrightarrow, \prec, \prec_m, \approx\}$ Str$_4$

Inference Rules

10. If $\vdash \phi \to \psi$ and $\vdash \phi$, then $\vdash \psi$ MP

Notice that IdSt$_1$ and IdSt$_2$ from earlier are encapsulated as the right-to-left sides of the principles Str$_2$ and Str$_1$, respectively. Note also that Str$_3$ and Str$_4$ are just generalizations of structured propositions that the grounding principles entail with the minimal logic of identity in the background; the only reason that we couldn't derive the more general form is that, at least as of now, we don't have grounding principles for the other connectives, such as $\to$ (see Section 5 for more on such principles). As a result, this means that Russellian propositions, characterized by RP above, prove all the cases of propositional identity and non-identity stated in Theorem 3.1, and of course many more. In other words, MPPI is a strict fragment of RP. So we have:

Theorem 3.2. *The unrestricted propositional calculus with identity proves all theorems stated in Theorem 3.1. That is, the latter can be derived from* RP$\cup$UNIG

Note that this observation holds at the level of propositional logics of grounding, without quantifiers taken into account. A similar situation holds for the *higher-order* logic of immediate ground (Fritz, 2021), where many instances of a general, higher-order formulation of a schema representing Russellian propositions are entailed. Thus, as it was claimed before, these altogether suggest that the Russellian account of propositions is the most systematic account that captures all the structure that emerges from the principles of grounding. In the future sections, We will officially adopt RP as our logic of propositional identity to provide our semantics for the logics of grounding explored in this section.

3.3 Factive Ground

As was mentioned before, factive immediate grounding statements are just non-factive statements where the relata of the grounding relation are both true; similarly for mediate grounding. That is, we have:

- $\phi \prec_f \psi := (\phi \wedge \psi) \wedge (\phi \prec \psi)$
- $\phi \prec_{fm} \psi := (\phi \wedge \psi) \wedge (\phi \prec_m \psi)$

The unrestricted logic of factive immediate ground is as follows (see, e.g., Wilhelm, 2020b; Fritz, 2021, for these principles):

Unrestricted Factive Immediate Ground (UFIG)

1. $(\phi \prec_f \psi) \to (\phi \wedge \psi)$ FG$_f$

2. $\phi \nprec_f \phi$ IG$_f$

3. $\big(\phi \prec_f (\psi \wedge \gamma)\big) \leftrightarrow ((\psi \wedge \gamma) \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ CG$_f$

4. $\big(\phi \prec_f (\psi \vee \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ DG$_f$

5. $\big(\phi \prec_f \neg(\psi \wedge \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma)))$ NCG$_f$

6. $\big(\phi \prec_f \neg(\psi \vee \gamma)\big) \leftrightarrow \big(\neg(\psi \vee \gamma) \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))\big)$ NDG$_f$

7. $\big(\phi \prec_f \neg\neg\psi\big) \leftrightarrow (\phi \wedge (\phi \approx \psi))$ NG$_f$

The unrestricted logic of factive mediate ground is as follows:

Unrestricted Factive Mediate Ground (UFMG)

1. $(\phi \prec_{fm} \psi) \to (\phi \wedge \psi)$ FG$_{fm}$

2. $\phi \nprec_{fm} \phi$ IG$_{fm}$

3. $((\phi \prec_{fm} \psi) \wedge (\psi \prec_{fm} \theta)) \to (\phi \prec_{fm} \theta)$ TRG$_{fm}$

4. $(\phi\wedge\psi) \to ((\phi \prec_{fm} (\phi\wedge\psi)) \wedge (\psi \prec_{fm} (\phi\wedge\psi)))$ CG$_{fm}$

5. $(\phi \to (\phi \prec_{fm} \phi\vee\psi)) \wedge (\psi \to (\psi \prec_{fm} \phi\vee\psi))$ DG$_{fm}$

6. $(\neg\phi \to (\neg\phi \prec_{fm} \neg(\phi\wedge\psi))) \wedge (\neg\psi \to (\neg\psi \prec_{fm} \neg(\phi\wedge\psi)))$ NCG$_{fm}$

7. $\neg(\phi\vee\psi) \to ((\neg\phi \prec_{fm} \neg(\phi\vee\psi)) \wedge (\neg\psi \prec_{fm} \neg(\phi\vee\psi)))$ NDG$_{fm}$

8. $\phi \to (\phi \prec_{fm} \neg\neg\phi)$ NG$_{fm}$

It is easy to check that the principles of the factive logics are just theorems of their corresponding non-factive logics. That is, we have the following, where $\vdash$ stands for or basic propositional logic in the background (PC), and $\bigwedge$ conjuncts all the principles of the relevant logics.:

Theorem 3.3. NFIG $\vdash \bigwedge$ UFIG *and* NFMG $\vdash \bigwedge$ UFMG.

Since these are straightforward (they only depend on the definition of 'factive ground' in terms of 'non-factive ground'), we won't provide proofs.

4 Semantics

We start by introducing our Russellian propositions, which, as expected, perfectly mirror the structures of the sentences of our language (hereafter we sometimes drop the qualification 'Russellian'). We use capital letters $P_1, P_2, ...$ to represent atomic *propositions*, as it were, and $A, B, C, ...$ as metavariables for general propositions. We also make bold the connectives; thus, e.g., we have the propositional connective $\boldsymbol{<}$ instead of the sentential connective $<$.[11]

Similar to our language, we assume to have infinitely many *atomic propositions* $P_1, P_2, ...$, and set the set of all of them with AT^*. Here's the definition of our propositions:

Definition 4.1 (Propositions)**.** *Propositions* are constructed as follows:

1. P_i is a proposition, where $i \in \mathbb{N}$,
2. If A and B are propositions, then so are $\neg A$ and $A \circ B$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, <, <_p, \approx\}$.

We signify the set of all propositions constructed in this way with $\mathcal{D}_{\langle\rangle}$, and call it the *propositional domain.*[12][13]

A crucial notion that plays role in our semantics of ground is that of grounding constituency. As we noticed in the previous section, logics of ground display certain structural patterns; grounding constituency encapsulates these patterns in their most general forms.

Definition 4.2 (Grounding Constituency)**.** We define *grounding constituency* as the relation $\sqsubset$ on the propositional domain $\mathcal{D}_{\langle\rangle}$, such that $A \sqsubset B$ (read: A is a grounding constituent of B) if, and only if, one of the following holds:

1. $B = A \circ C$ for some $C \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,
2. $B = C \circ A$ for some $C \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,

[11] In effect, we could use our object language $\mathcal{L}$ in a new capacity now, but to avoid potential confusion, we proceed as in here.

[12] Note that in both the definition of 'language' and 'proposition' we are constructing the relevant entities recursively, with the implicit assumption that a pair of sentences or propositions are identical if and only if they have the same structure. This really is enough in laying out the idea of unique readability, as is common in logic as well as philosophy texts. That said, the idea of unique readability can be more explicitly encoded in both Definitions 3.1 and 4.1 by using the notion of sets and set identity; we avoid this in the interest of simplicity. For example, one can take an approach along the following lines: first, take the P_is (atomic propositions) and all the propositional connectives ($\wedge, <$, etc.) as constituting an appropriate set of pairwise distinct sets (one suitable choice may be this: for each $i \in \mathbb{N}$, define $P_i := \mathbb{N} \times i$, $\neg := \mathbb{R}$, $\wedge := \mathbb{R}^2$, $\vee := \mathbb{R}^3$, $< := \mathbb{R}^4$, ...). Then, define the structured propositions using tuples—e.g., $A \circ B$ as $(A, \circ, B)$ and $\neg A$ as $(\neg, A)$. From this, all the Russellian propositions follow. For example, it follows that, e.g., $A \circ B = C \circ D$ if and only if $A = B$ and $C = D$, where $=$ is set identity. It also follows that no negative proposition of the form $\neg A$ can be identical to a composite proposition of the form $B \circ C$, that is $\neg A \neq (B \circ C)$; similarly, no proposition is identical to its double negation, i.e., $A \neq \neg\neg A$, and no proposition is identical to its composition with another proposition, i.e., $A \neq (A \circ B)$ and $A \neq (B \circ A)$.

[13] Our models assume that we have infinity many distinct atomic propositions, which is a reasonable assumption, e.g., within a richer language that accommodates unary predicates and under the common view that any sentence of the form $F(a)$, where F is a unary, non-logical, predicate and a is an object, is an atomic proposition. There is, however, nothing crucial in what follows that hinges on this assumption; a finite base of atomic propositions will equally do and grant unique readability.

3. $A = \neg A^* \ \& \ B = \neg(A^* \circ B^*)$, for some $A^*, B^* \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,

4. $A = \neg A^* \ \& \ B = \neg(B^* \circ A^*)$, for some $A^*, B^* \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,

5. $B = \neg\neg A$.

As was mentioned before, to capture mediate grounding, we can think of statements of mediate grounding as obtained through 'chaining' immediate grounding statements. To implement this idea into our semantics, we define $\sqsubset^*$ as the transitive closure of $\sqsubset$.[14][15]

Here's how we specify the truth value of our propositions:

Definition 4.3 (Atomic Truth Function; Truth Function)**.** An *atomic truth function* is a function $at : AT^* \to \{0, 1\}$. We define the *truth function* based on any atomic truth function at as the function $T_{at} : \mathcal{D}_{\langle\rangle} \to \{0, 1\}$ based on at as follows:

1. $T_{at}(P_i) = at(P_i)$, if and only if $P_i \in AT^*$

2. $T_{at}(\neg A) = 1$, if and only if $T_{at}(A) = 0$

3. Other Booleans as usual

4. $T_{at}(A \approx B) = 1$, if and only if $A = B$

5. $T_{at}(A \prec B) = 1$, if and only if $A \sqsubset B$

6. $T_{at}(A \prec_m B) = 1$, if and only if $A \sqsubset^* B$

Notice that the truth values of complex propositions are sensitive to the truth value of their constituents only in the case of Boolean propositions; what determines the truth value of identity and grounding statements is only the structure of the constituent propositions involved.

[14]The transitive closure of a binary relation R on a set X, in general, is the smallest relation R^* on X that contains R and is transitive, i.e., if aR^*b and bR^*c, then aR^*c. It's easy to prove that every binary relation has a transitive closure.

[15]Note that here we're not defining mediate grounding ($\prec_m$) as the transitive closure of immediate grounding ($\prec$) at the level of object language; we're merely *interpreting* the former as the transitive closure of the latter's interpretation, that is interdefining them at the level of semantics. The connectives themselves are treated as primitives in this paper, so aren't to be interdefined (see Definition 3.1). One might suggest that if we interdefine the connectives themselves, e.g., $\prec_m$ in terms of $\prec$, using the notion of transitive closure, the principles of UNMG will presumably just follow from those of UNIG. That might be true, but it would require higher-order quantification tools which are unavailable in our propositional language. For instance, one might define $\prec_m$ as follows: $\phi \prec_m \psi := \exists n \in \mathbb{N}(\phi = \phi_1 \prec \phi_2 \prec ... \prec \phi_n = \psi$. Alternatively, one might suggest just embracing the transitivity of mediate ground (i.e., TRG_m), and the rest of the principles of UNMG follow from that and UNIG. But even if some of the principles of UNMG follow, not all will—IG_m is an example. As for what *is* really the relationship between $\prec_m$ and $\prec$, i.e., of immediate and mediate grounding at the level of the object language, the answer seems unclear while we have no quantificational tools at our disposal. Our goal in this paper is to find a semantics for the principles of immediate and mediate ground, as entertained in the literature, and their interdefinability at the level of semantics does that for us. Thanks to an anonymous referee for drawing my attention to this issue.

We now introduce our semantics by defining the 'interpretations' of statements of our language, which are essentially the propositions they denote.

Definition 4.4 (Assignment Function; Interpretation)**.** An *assignment function* is a function of the form $a : AT \to \mathcal{D}_{\langle\rangle}$. For any such function we define the *interpretation* based on a as the function $[\![.]\!]_a : \mathcal{L} \to \mathcal{D}_{\langle\rangle}$, such that:

1. $[\![p_i]\!]_a = a(p_i)$, if $p_i \in AT$,
2. $[\![\neg\phi]\!]_a = \boldsymbol{\neg}[\![\phi]\!]_a$,
3. $[\![\phi \circ \psi]\!]_a = [\![\phi]\!]_a \boldsymbol{\circ} [\![\psi]\!]_a$, where $\circ \in \{\wedge, \vee, \to, \leftrightarrow, \prec, \prec_m, \approx\}$ and $\boldsymbol{\circ}$ is the corresponding (emboldened) propositional operator.

We call any triple $(\mathcal{D}_{\langle\rangle}, at, a)$ a *propositional model*, where $\mathcal{D}_{\langle\rangle}$ is our propositional domain, at is an atomic truth function and a an assignment function. For a model $M := (\mathcal{D}_{\langle\rangle}, at, a)$ and sentence $\phi \in \mathcal{L}$, we say that ϕ is *true with respect to* M, written $M \vDash \phi$, if $T_{at}([\![\phi]\!]_a) = 1$. We call ϕ *valid* or a *truth*, written $\vDash \phi$, if it's true with respect to every model.[16]

Notice that in our models it's possible to assign *any* proposition whatsoever to any sentential letter of the language. This marks an important difference between our semantics with the one in Correia (2017): in the latter, 'crucially', sentential variables of the language cannot be assigned complex propositions. As Correia himself notes (see p. 517), this 'unorthodoxy' has a bearing on the applications of his logic to statements of natural language, and thus 'care is needed in order to apply the logic'.[17]

In any case, all of our logics of grounding, i.e., the unrestricted logic of immediate and mediate logics, both non-factive and factive, as well as Russellian Propositions, are sound with respect to our semantics (see Appendix II for a proof). Suppose $\vdash$ stands for derivability from UNIG ∪ UNMG ∪ UFIG ∪ UFMG ∪ RP.

Theorem 4.1 (Soundness)**.** *If* $\vdash \phi$*, then* $\vDash \phi$*.*

[16]Note that from the semantics it follows that atomic propositions cannot ground one another. In particular, one might think "the fact that my shirt is maroon grounds the fact that it is red" (e.g., see Audi, 2012, p. 693), but our semantics doesn't accommodate that. This might be considered as a shortcoming of the semantics, however, our concern here is the logic of *impure* ground, with certain standard principles in mind. To the author's best knowledge, none of the alternative semantics in the literature (each imposing some kind of structure on propositions) can accommodate such claims, so even if they are correct, this won't be a unique problem to our semantics. Moreover, one might be able to somehow enrich the current semantics in a language where non-logical predicates are allowed, and take into account the inter-definability of properties in accounting for grounding statements containing them (Kiani, MSa, does this in a rigorous way for the neighboring notion of entity grounding)

[17]To give an example similar to Correia's: in natural languages, we can have the sentence 'Pluto is grue' (call it ϕ) to express the disjunctive proposition that Pluto is green or Pluto is blue. Assuming that the first and second disjuncts are respectively expressed by 'Pluto is green' (ψ) and 'Pluto is blue' (γ), we would normally want to consider the formal statements $\psi \prec \phi$ and $\gamma \prec \phi$ as true, but there's no way to get Correia's semantics to validate this judgment.

It is worth noticing that, as the proof of this theorem (as well as other soundness results from the next section) shows (see Appendix II), the fact that we are working with Russellian propositions plays a crucial role in our results.

Now, consider, e.g., the assignment function a such that $a(p_i) = P_i$ for all $p_i \in AT$, and the truth function at such that $at(P_i) = 0$. (Or consider any other pairs of assignment and truth functions, for that matter.) By soundness, the induced model validates all the axioms of our logic. So, we have:

Corollary 4.1 (Consistency)**.** *The unrestricted propositional logics of immediate and mediate ground with identity, both non-factive and factive, are consistent. In other words,* UNIG ∪ UNMG ∪ UFIG ∪ UFMG ∪ RP *is consistent.*

Notice that our semantics can capture the infinitely many theorems of the kinds below which follow from our unrestricted logics, but which, as was noted in Section 2, the existing semantics in the literature fail to capture:

1. $(\phi \prec \psi) \prec ((\phi \prec \psi) \wedge \psi)$
2. $(\phi \approx \phi) \prec ((\phi \approx \phi) \vee \psi)$
3. $\big((\phi \approx \phi) \wedge \psi\big) \prec \big((\phi \vee \phi) \vee ((\phi \approx \phi) \wedge \psi)\big)$
4. $\Big((\phi \approx \phi) \prec ((\phi \approx \phi) \wedge \psi)\Big) \nprec \Big((\phi \approx \phi) \prec ((\phi \approx \phi) \wedge \psi)\Big)$

$\vdots$

These are declared as true statements by our semantics, simply because $[\![\phi \prec \psi]\!]_a \sqsubset [\![(\phi \prec \psi) \wedge \psi]\!]_a$, $[\![\phi \approx \phi]\!]_a \sqsubset [\![(\phi \approx \phi) \vee \psi]\!]_a$ and so on, for any assignment function a.

We conclude the section by shedding light on the question of completeness. It's straightforward to see that our logics are not complete with respect to the proposed semantics. For instance, given the definition of grounding constituency, no model can validate a grounding statement where the groundee is itself a grounding or identity statement, as they don't have the right structure to enter into the grounding constituency relation. That is, any statements of the form $\phi \prec (\psi \prec \gamma)$ or $\phi \prec (\psi \approx \gamma)$ is falsified by all models, no matter what propositions the schematic letters stand for; so their negations $\phi \nprec (\psi \prec \gamma)$ and $\phi \nprec (\psi \approx \gamma)$ must be valid. But there are no principles in our logic that would prove such statements.

At this point, one can choose between two options to achieve completeness: (i) add certain principles such as the ones above to the logic and make up for the gap, or (ii) extend the notion of grounding constituency in a way that, e.g., a proposition like A is considered as a grounding constituent of certain propositions like $C \boldsymbol{\prec} D$ and $C \boldsymbol{\approx} D$, thus avoiding the gap in a different way. (In the next section, we discuss some extensions along both lines in more detail.)

To be clear, although both of these options lead to filling some gap between our logic and semantics, that may or may not lead to completeness; we leave open how the gap is to be fully closed, and hence completeness achieved. However, we conclude the section by stating a close result, stating that our semantics, in its current form, gets right all the positive grounding claims (i.e., those that aren't in forms of negation); for a proof of this see Appendix II.

Theorem 4.2. *Every positive grounding and identity truth is provable:*

(i) If $\vDash \phi \prec \psi$*, then* $\vdash \phi \prec \psi$*,* *(ii) If* $\vDash \phi \prec_f \psi$*, then* $\vdash \phi \prec_f \psi$*,*

(iii) If $\vDash \phi \prec_m \psi$*, then* $\vdash \phi \prec_m \psi$*,* *(iv) If* $\vDash \phi \prec_{fm} \psi$*, then* $\vdash \phi \prec_{fm} \psi$*,*

(v) If $\vDash \phi \approx \psi$*, then* $\vdash \phi \approx \psi$*.*

5 Extensions

We now discuss two kinds of desirable extensions of our logics and semantics that aren't available to the existing semantic projects (Krämer, 2018; Correia, 2017; deRosset and Fine, 2022).[18]

5.1 Grounds of other Boolean Statements

There aren't many works on grounds of Boolean statements other than those that only contain instances of conjunction, disjunction and negation. An exception to this is Schnieder (2011), where, in laying down certain rules governing the logic of 'because' he proposes (the factive versions of) most of the rules that we have listed before under the factive logic of mediate ground, plus other Boolean cases. For instance, he offers the following principles regarding the grounds of conditional statements:

- $(\neg\phi \prec_m (\phi \to \psi)) \wedge (\psi \prec_m (\phi \to \psi))$ CoG$_m$
- $(\phi \prec_m \neg(\phi \to \psi)) \wedge (\neg\psi \prec_m \neg(\phi \to \psi))$ NCoG$_m$

As expected, the following are the corresponding immediate principles:

- $\gamma \prec (\phi \to \psi) \leftrightarrow (\gamma \approx \neg\phi \vee \gamma \approx \psi)$ CoG
- $\gamma \prec \neg(\phi \to \psi) \leftrightarrow (\gamma \approx \phi \vee \gamma \approx \neg\psi)$ NCoG

To accommodate this, we can simply extend the notion of grounding constituency in a way that the desired principles of conditional grounding are accommodated. More specifically, we can add the following four clauses to the definition of $A \sqsubset B$ (Definition 4.2):

- $A = \neg A^*$ and $B = A^* \to C$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$,

[18] See Poggiolesi and Francez (2021) for a tentative logic of 'exclusive' and 'ternary' notions of disjunction. While it is likely that these notions can also be captured in our approach as well, we don't attempt to establish that in this paper, as this paper is focused on the more urgent and widely used connectives that lack expressive semantics as shown.

- $B = C \rightarrow A$ for some $C \in \mathcal{D}_{\langle\rangle}$,
- $A = \neg A^*$ and $B = \neg(C \rightarrow A^*)$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$,
- $B = \neg(A \rightarrow C)$ for some $C \in \mathcal{D}_{\langle\rangle}$

Note that the semantics of ground expanded in this way also captures the logic of 'because' in Schnieder (2011), and in particular proves its consistency.[19] In general, assuming that we extend our models to accommodate the extended notion of grounding constituency, we have the following (see Appendix II for a proof):

Theorem 5.1. *CoG and NCoG are both valid.*

Consequently, the extended logics are all consistent as well.

Before moving on to the next type of extension, a remark is in order. Note that in stating the definition of our language $\mathcal{L}$ (Definition 4.1), we treated all the connectives from $\{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ as primitives, thus, in particular, avoided interdefining any of Boolean statements in terms of other ones. One might suggest otherwise, to deduce the relevant grounding principles from a smaller set of principles. For instance, it might be suggested to interdefine $\phi \rightarrow \psi$ as, e.g., $\neg\phi \vee \psi$ and deduce the principles of conditional grounding laid out above in terms of the principles of disjunctive grounds.

But this can't be easily done. To get a sense of complications attached to such identifications, suppose the identity above holds, thus $\phi \rightarrow \psi \approx \neg\phi \vee \psi$ is a theorem of our background identity logic. It then follows from DG and $\textsc{Str}_1$ that $\neg\phi \prec \phi \rightarrow \psi$ and $\psi \prec \phi \rightarrow \psi$. So far, so good: these in fact follow from CoG. But note that by NCoG, we have $\phi \prec \neg(\phi \rightarrow \psi)$. So, since from $\phi \rightarrow \psi \approx \neg\phi \vee \psi$ we have $\neg(\phi \rightarrow \psi) \approx \neg(\neg\phi \vee \psi)$, by $\textsc{Str}_1$ we have $\phi \prec \neg(\neg\phi \vee \psi)$. However, from NDG applied to $\neg(\neg\phi \vee \psi)$ it follows that the immediate grounds of $\neg(\neg\phi \vee \psi)$ are *only* $\neg\neg\phi$ and $\neg\psi$, hence, given that $\phi \not\approx \neg\neg\phi$, we must have $\phi \approx \neg\psi$. In other words, for any conditional $\phi \rightarrow \psi$ where $\phi \not\approx \neg\psi$, the identification of $\phi \rightarrow \psi$ with $\neg\phi \vee \psi$ leads to the inconsistency of extensions of our logical system with the plausible principles of immediate conditional ground due to Schnieder (2011).

In response to this, one might suggest rejecting one of the principles of, e.g., conditional grounding. But why do so? They are no less plausible than those governing conjunction or disjunction (also, see footnote 16 for a unified motivation behind them all). More importantly, as we noted above, the extended logic *is* provably consistent. So, the better option seems to be one that leaves the connectives alone and avoids reducing them to one another.

[19]The general guiding principle behind Schnieder's logic is called 'core intuition', which he defines as follows (p. 448): 'A sentence governed by a classical truth-functional connective has its truth value because of the truth values of the embedded sentences.' The other Boolean cases dismissed here can be accounted for in a similar manner to the case of conditional grounding.

5.2 Iterated and Identity Grounding

We mentioned in Section 2 that, according to some authors (e.g., Bennett, 2011), a grounding fact like $\phi \prec \psi$ is grounded by its ground ϕ. If we take the immediate ground of $\phi \prec \psi$ to be only ϕ, then we have the following:

- $\gamma \prec (\phi \prec \psi) \leftrightarrow \gamma \approx \phi$ IDG

We also mentioned another plausible principle regarding grounds of statements of propositional identity: according to Wilhelm (2020a), e.g., identity statements of the general form $a \approx a$ are ('entity-')grounded by a, where a can be any entity. Someone might pick up this idea and argue for a fact-grounding counterpart of it, along the following lines (see footnote 5 for a proviso):

- $\psi \prec (\phi \approx \phi) \leftrightarrow \psi \approx \phi$ GG

Again, we can revise the notion of grounding constituency in a way that this is accounted for in our semantics, by adding the following clauses to the definition of $A \sqsubset B$ (Definition 4.2):

- $B = A \boldsymbol{\prec} C$ for some $C \in \mathcal{D}_{\langle\rangle}$,
- $B = A \boldsymbol{\approx} A$

Suppose we extend our conception of models to accommodate the extension of grounding constituency with these. Then we have the following (the proof is as straightforward as previous cases, so we omit them):

Theorem 5.2. *DIG and GG are both valid.*

As usual, the extended logics turn out to be consistent too.

5.3 Extending the Existing Semantics in the Literature

We conclude the section by reflecting on the status of the existing semantics (found in Correia, 2017; Krämer, 2018; deRosset and Fine, 2022) with regards to extensions of the logics and semantics along the lines above. We briefly noted in Section 2 that these semantics fail to accommodate an infinitude of grounding facts that follow from unrestricted impure logics of ground, as well as the distinct views on identity and iterated grounding glossed above, due to the artificial restrictions imposed on the languages in which the logics are formulated, where statements or propositions of iterated and identity grounding aren't allowed in the relata of ground.

Can these semantics be revised, though, to make up for these shortcomings? I charitably assume that *given the same linguistic limits*, all these semantics can be extended without any trouble, to capture the other Boolean extensions of their logics and semantics (though this really depends on certain details at play, I ignore that).

But what about the extensions that lift the restrictions of the language and logics to allow for statements of iterated and identity grounding to appear in the relata of grounding statements? In the case of Correia (2017), where he works with structured propositions of some sort, it might be possible to make certain revisions and generalizations to allow for the semantics to accommodate the unrestricted version of the principles he works with, though it's not clear if the semantic results in the paper that heavily rely on these notions stay intact under such extensions.[20] I leave this issue open here.

However, unlike Correia (2017), both Krämer (2018) and deRosset and Fine (2022) work with the *truthmaker* content of propositions. The idea of applying truthmaker semantics to logics of ground goes back to Fine (2012b), where he provides an elegant, sound and complete semantics for the pure logic of ground in terms of truthmakers. The original semantics of Fine (2012b) takes the semantic value of a statement to be its set of 'verifiers', i.e., the set of 'states' or 'facts' that make true the statement.

But while truthmakers work perfectly well in the case of pure logic, the impure logic soon displays various forms of resistance to the plain truthmaker semantics that Fine works with (see Fine, 2012a, footnote 22 for some early notes on this issue). For example, in order for the principles NG and IG to both go through, we need the truthmaker content of a sentence and its double negation to be, at the very least, distinct. Truthmaker semantics doesn't provide this: what verifies $\neg\phi$ falsifies ϕ, and what falsifies $\neg\phi$ verifies ϕ. So what verifies $\neg\neg\phi$ verifies ϕ, and what falsifies $\neg\neg\phi$ what falsifies ϕ; so ϕ and $\neg\neg\phi$ have the same truthmaker content, hence are identical. A similar problem holds for disjunction and conjunction: due to the standard way their truthmaker contents are defined, it turns out that ϕ, $\phi \vee \phi$, and $\phi \wedge \phi$ also have the same content, so the relevant instances of the principles CG and DG fail to hold.

So, to make truthmaker semantics work, certain revisions must be made on its standard workings. In particular, some sort of structure must be imposed on the truthmaker content of sentences so that, e.g., ϕ, $\neg\neg\phi$, $\phi\vee\phi$ and $\phi\wedge\phi$ have pairwise distinct truthmaker contents; once that's done, the semantics of grounding must be given in a way that the desired principles of the impure logics are accounted for. This is exactly what both Krämer (2018) and deRosset and Fine (2022) do, though each in their own way. Krämer (2018) designs his 'mode-ified' semantics of ground, where his semantics relies on the 'modes of verification' of sentences—something which, according to himself, 'corresponds to a certain kind of answer to the question of how a truth is verified by a fact' (p. 786), and, in any case, leads to the required distinctions of truthmaker contents.

deRosset and Fine (2022), on the other hand, do a deeper dive into the se-

[20] The core notions upon which models are built in Correia (2017)—e.g., that of 'propositional structure' (which is the space of propositions and operators that construct them), 'degree' (measuring the complexity of propositions) and 'grounding relation' (holding between members of the propositional structure) (see pp. 518-19)—simply exclude cases of the sort where the relata of grounding are grounding or identity propositions. It's not clear which, if any, of the semantic results that heavily rely on these restrictions, such as soundness and completeness, can be retained under relevant extensions of these notions.

mantics of a particular system of ground, called System GG, closely related to the logical system originally proposed in Fine (2012a). The semantics that this work proposes is both sound and complete, and captures total, as well as, partial grounds. It is also, like the semantics in Krämer (2018), based on a revised form truth-makers semantics which accommodates the appropriate structure that propositions need for the relevant principles of ground to go through. More specifically, deRosset and Fine adopt two notions of fusion that are more fine-grained than the usual one—'combination' (for conjunction) and 'choice (for disjunction)—and somehow elegantly capture the structured principles of System GG in a sound and complete way.

But as nice as the semantics in deRosset and Fine (2022) is in comparison to the other works in the literature, it still suffers from the exact same issue that the previous semantics do: the language is limited to the usual Boolean connectives (see, e.g., pp. 12-13 of the mentioned paper) and the semantics is designed to exactly capture that, and nothing else. It's not clear when we extend the language, the semantics will be able to catch up.

In fact, aside from the artificial limitations of language and how that is built into the semantics, which is what all the existing semantics suffer from, there is a more profound issue with the semantics that particularly work with truthmakers. The issue is that even though it is straightforward to determine the verifiers and falsifiers of Boolean statements in whatever level of granularity, using truthmakers (in terms of 'fusions', 'manners of fusions', 'combination' or 'choice' of states; see, e.g., Fine, 2017c,a,b; Krämer, 2018; deRosset and Fine, 2022), it is, in general, not clear at all how to account for the verifiers and falsifiers of *grounding* or *identity* statements in terms of such fusions. This would require granting access to the truthmaker content of identity or grounding statements, and that's where truthmaker semantics hits the bottom.

In general, truthmaker semantics is in its infancy, and, to the best of our knowledge, there just isn't any work in the literature that would tell us what the truthmaker contents of statements other than Boolean, quantificational or modal statements look like—certainly not the truthmaker content of grounding or identity statements. And this problem doesn't seem to be easily resolvable: due to the hyperintensionality of ground and identity, it's very unlikely for the truthmaker content of grounding or identity statements to be definable using the truth-functionally behaved operation of 'fusion' on states, no matter how granular they are manipulated to become. Further operations on state spaces are likely required, and, as of now, it's not clear what they would look like or if they can be philosophically motivated or formally developed in a plausible and consistent way.

The semantics that we provided in this paper, however, has in its premise the extreme flexibility that any extension of the language can ever want, because it simply reflects the structure of sentences in the language to the semantics, using the idea of Russellian propositions. Whatever operator one adds to the language will be mirrored to the semantics; all it takes for the semantics to capture the logic of the newly added connectives is to simply revise the definition of 'Grounding

Constituency', in the way that was shown in this section.

6 Conclusion

I showed that models for sentence-like, Russellian propositions can be used to provide a unified, simple and highly expressive semantics for various unrestricted propositional logics of ground. I also showed that our semantics can be extended to accommodate certain distinct philosophical positions about grounds of grounding and identity statements. We noted that the existing semantics in the literature (Krämer, 2018; Correia, 2017; deRosset and Fine, 2022) fail to do either of these. More importantly, even though we left it open whether Correia's semantics is safely extendable to accommodate these, we noted that the prospects of extending the semantics in Krämer (2018) and deRosset and Fine (2022), and in general, any truthmaker semantics of ground is bleak and dependent on unexplored limitations of truthmaker semantics. Moreover, even though formally satisfying (within the usual artificial boundaries of the language that my paper conveniently surpasses), it's not clear how satisfying these revisions of the truth-maker semantics are at a conceptual level, and how such diversity of truth-maker semantics found in the literature (each serving a specific philosophical purpose) and the levels of content granularity that follow from them can be summed up and explained in a bigger picture.

Also, aside from the expressiveness and predictable high flexibility of our semantics, there are certain advantages of our project over those that assume less granular accounts of propositions in accounting for the semantics of logics of ground (as in Correia, 2017) or addressing their consistencies (as in Wilhelm, 2020b). For one, our assumption of grain is much more systematic and widely entertained in the literature, ranging from attitude contexts in the philosophy of language (as in, e.g., Kaplan, 1989 [1977]; King, 1996, 2009; Soames, 1987), to the neighboring notions of metaphysical priority, essence and ontological dependence (as in, e.g., Fine, 1995, 1980, 1994). Moreover, we noted that the Russellian view is arguably the most straightforward and systematic account of propositions that explains all the built-in structural commitments of the notion of ground explored in this paper and in Fritz (2021).

But as popular and useful as Russellian propositions are, basic Cantorian reasoning about cardinalities reveals that their assumption leads to certain inconsistencies—an issue that, interestingly, was first mentioned in the original work of Russell (1903) himself (see Appendix B), and later was re-discovered by Myhill (1958) (hence, the *Russell-Myhill* paradox), but surprisingly has been completely ignored in most recent works that assume or argue for Russellian propositions, as cited earlier (see Deutsch, 2008, on this ignorance and its consequences for philosophy). In fact, only recently has this paradox been rediscovered within the background of simple type theory, thus bringing to light the inconsistency of Russellian propositions with the standard assumptions of higher-order logic (see, e.g., Hodes, 2015; Goodman, 2017). Accordingly, this might be taken to undermine the conceptual value of our project, suggesting that the models to be presented are merely mathematical constructs that by no means represent propositions.

The situation becomes even more dramatic when we realize that, as Fritz (2021) notes, the instances of Russellian propositions that the higher-order logic of immediate ground entails happen to be sufficient to reconstruct the Russell-Myhill result, effectively establishing the inconsistency of the relevant higher-order logic of ground in question. In other words, not every proposition has to be Russellian for the paradox to go through—a certain, smaller fragment of propositions that are so is sufficient to reconstruct the paradox. The higher-order logic of ground gives us just one such fragment and hence is inconsistent.

This portrays a rather bleak picture of the notion of ground when coupled with considerations of granularity, and leaves one wondering if there's a way to save logics of ground from the troubles of grain.

In response to these issues, in a series of broadly related papers I adopt a picture of propositions (as well as other relational entities, such as properties and relations), reminiscent of Russell (1908); Whitehead and Russell (1910), according to which they come in infinite levels, in a way that roughly put, the inhabitants of higher levels are systematically obtained through quantification over the ones from lower levels. We call this view the *ramified* account of propositions (similarly for other relational entities). Once the ramified picture is deployed, one can consistently reformulate the Russellian view and avoid the Russell-Myhill paradox, as well as ground's higher-order inconsistency result due to Fritz.[21]

Aside from establishing the consistency of the ramified Russellian theory of propositions (explored in Kiani, MSc), one aim of the series is to show that this view can itself be independently motivated via certain considerations having to do with a neighboring notion of metaphysical priority, namely 'entity grounding' (as introduced in, e.g., Wilhelm, 2020a; Schaffer, 2009; deRosset, 2013); this is shown in Kiani (MSa). Another part of the series shows that ramified Russellian propositions can be leveraged to semantically account for and establish the consistency of various logics of ground—ranging from propositional to higher-order—and provide a unified 'predicative' solution to a cluster of paradoxes of quantificational ground that have emerged in recent years (e.g., Donaldson, 2017; Fine, 2010; Krämer, 2013); a type of solution that has long been predicted but remained fairly underexplored (see, e.g., Fine, 2010; Krämer, 2013; Korbmacher, 2018b,a, for various forms of these puzzles and some solutions to certain variants of them).

It is this latter part with which the present paper was concerned: while higher-order logics of ramified ground, their semantics and consistency results, as well as their contribution to puzzles of quantificational ground, are all explored in Kiani (MSb), the task of this paper was to only explore how the assumption of Russellian propositions alone can be leveraged to semantically account for vari-

[21]Other solutions to the Russell-Myhill paradox can be given that are not based on ramified types. For instance, Deutsch (2014) proposes a solution that is based on set theory. But it's not clear how, aside from their mathematical use, such solutions fare in the context of grounding and, in general, metaphysics; we leave that open here.

ous *propositional* logics of ground and establish their consistencies. The semantic contributions of this paper prepare the groundwork based on which the more sophisticated, higher-order logics of ground from Kiani (MSb) are semantically accounted for. As such, since here we only treat propositional logics of ground without quantification, implementing ramification isn't needed, and the assumption of Russellian propositions suffices for our purposes. As we have noticed, none of the results obtained in this paper rely on the other works in the series.

I left the questions of completeness open. Also, throughout the paper, I've only focused on strict partial grounds and their logics. As a result, other variants of grounding relations, such as total and weak grounding, as well as their logics, still need to be semantically accounted for. I wish to attend to these issues in the future.

Appendix I: Logics of Identity and Ground

Here we repeat all the principles of grounding, as well as propositional identity, that we explored in the paper, for their accessibility and use in the formal proofs in the next appendix.

MINIMAL PRINCIPLES OF PROPOSITIONAL IDENTITY (MPPI)

1. $\phi \approx \phi$ — REF
2. $(\phi \approx \psi) \to (\psi \approx \phi)$ — SYM
3. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \to (\phi \approx \gamma)$ — TR
4. $((\phi \approx \psi) \wedge \phi) \to \psi$ — IDTR
5. $(\phi \approx \psi) \to (\neg\phi \approx \neg\psi)$ — IDST_1
6. $((\phi \approx \psi) \wedge (\gamma \approx \theta)) \to ((\phi \circ \gamma) \approx (\psi \circ \theta))$, where $\circ \in \{\wedge, \vee, \to, \leftrightarrow, \prec, \prec_m, \approx\}$ — IDST_2

RUSSELLIAN PROPOSITIONS (RP)

Axioms

1. Theorems of propositional calculus — PC
2. $\phi \approx \phi$ — REF
3. $(\phi \approx \psi) \to (\psi \approx \phi)$ — SYM
4. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \to (\phi \approx \gamma)$ — TR
5. $((\phi \approx \psi) \wedge \phi) \to \psi$ — IDTR

6. $((\phi \circ \psi) \approx (\gamma \circ \theta)) \leftrightarrow ((\phi \approx \gamma) \wedge (\psi \approx \theta))$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ STR$_1$
7. $(\neg\phi \approx \neg\psi) \leftrightarrow (\phi \approx \psi)$ STR$_2$
8. $(\phi \circ_1 \psi) \not\approx (\gamma \circ_2 \theta)$, where $\circ_1 \neq \circ_2 \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ STR$_3$
9. $\neg\phi \not\approx (\psi \circ \gamma)$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ STR$_4$

Inference Rules

10. If $\vdash \phi \rightarrow \psi$ and $\vdash \phi$, then $\vdash \psi$ MP

UNRESTRICTED NON-FACTIVE IMMEDIATE GROUND (UNIG)

1. $\phi \nprec \phi$ IG
2. $(\phi \prec (\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ CG
3. $(\phi \prec (\psi \vee \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ DG
4. $(\phi \prec \neg(\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ NCG
5. $(\phi \prec \neg(\psi \vee \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ NDG
6. $(\phi \prec \neg\neg\psi) \leftrightarrow (\phi \approx \psi)$ NG

UNRESTRICTED NON-FACTIVE MEDIATE GROUND (UNMG)

1. $\phi \nprec_m \phi$ IG$_m$
2. $((\phi \prec_m \psi) \wedge (\psi \prec_m \theta)) \rightarrow \phi \prec_m \theta$ TRG$_m$
3. $(\phi \prec_m (\phi \wedge \psi)) \wedge (\psi \prec_m (\phi \wedge \psi))$ CG$_m$
4. $(\phi \prec_m (\phi \vee \psi)) \wedge (\psi \prec_m (\phi \vee \psi))$ DG$_m$
5. $(\neg\phi \prec_m \neg(\phi \wedge \psi)) \wedge (\neg\psi \prec_m \neg(\phi \wedge \psi))$ NCG$_m$
6. $(\neg\phi \prec_m \neg(\phi \vee \psi)) \wedge (\neg\psi \prec_m \neg(\phi \vee \psi))$ NDG$_m$
7. $\phi \prec_m \neg\neg\phi$ NG$_m$

UNRESTRICTED FACTIVE IMMEDIATE GROUND (UFIG)

1. $(\phi \prec_f \psi) \rightarrow (\phi \wedge \psi)$ FG$_f$
2. $\phi \nprec_f \phi$ IG$_f$
3. $\big(\phi \prec_f (\psi \wedge \gamma)\big) \leftrightarrow ((\psi \wedge \gamma) \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ CG$_f$
4. $\big(\phi \prec_f (\psi \vee \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ DG$_f$

5. $\big(\phi \prec_f \neg(\psi \wedge \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma)))$ NCG$_f$

6. $\big(\phi \prec_f \neg(\psi \vee \gamma)\big) \leftrightarrow \big(\neg(\psi \vee \gamma) \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))\big)$ NDG$_f$

7. $\big(\phi \prec_f \neg\neg\psi\big) \leftrightarrow (\phi \wedge (\phi \approx \psi))$ NG$_f$

Unrestricted Factive Mediate Ground (UFMG)

1. $(\phi \prec_{fm} \psi) \to (\phi \wedge \psi)$ FG$_{fm}$
2. $\phi \nprec_{fm} \phi$ IG$_{fm}$
3. $((\phi \prec_{fm} \psi) \wedge (\psi \prec_{fm} \theta)) \to (\phi \prec_{fm} \theta)$ TRG$_{fm}$
4. $(\phi{\wedge}\psi) \to ((\phi \prec_{fm} (\phi{\wedge}\psi)) \wedge (\psi \prec_{fm} (\phi{\wedge}\psi)))$ CG$_{fm}$
5. $(\phi \to (\phi \prec_{fm} \phi{\vee}\psi)) \wedge (\psi \to (\psi \prec_{fm} \phi{\vee}\psi))$ DG$_{fm}$
6. $(\neg\phi \to (\neg\phi \prec_{fm} \neg(\phi{\wedge}\psi))) \wedge (\neg\psi \to (\neg\psi \prec_{fm} \neg(\phi{\wedge}\psi)))$ NCG$_{fm}$
7. $\neg(\phi{\vee}\psi) \to ((\neg\phi \prec_{fm} \neg(\phi{\vee}\psi)) \wedge (\neg\psi \prec_{fm} \neg(\phi{\vee}\psi)))$ NDG$_{fm}$
8. $\phi \to (\phi \prec_{fm} \neg\neg\phi)$ NG$_{fm}$

Appendix II: Technical Results

Proof (**Theorem 3.1**). We only give proof for all items where $\circ$ is $\wedge$; the cases where $\circ$ is $\vee$ can be proved quite similarly. In the cases where we prove the theorem by contradiction, we specify the assumption that is to be refuted in the end, for readability, and obtain a contradiction, $\bot$.[22]

- $\phi \not\approx \neg\neg\phi$

Proof.

(1)	$\phi \approx \neg\neg\phi$	Assumption (to be refuted)
(2)	$\phi \approx \phi$	Ref
(3)	$\phi \prec \neg\neg\phi$	NG
(4)	$(\phi \prec \phi) \approx (\phi \prec \neg\neg\phi)$	IdSt$_2$ 1, 2
(5)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \phi)$	Sym 4
(6)	$\phi \prec \phi$	IdTr 3, 5
(7)	$\phi \nprec \phi$	IG
(8)	$\bot$	PC 6, 7 □

- $\phi \not\approx (\phi \wedge \phi)$

[22] The proofs proceed in Hilbert-Style for higher rigor, where at each line the relevant axiom and potentially the previous lines or theorems are cited. One can offer an English reading of such proofs for higher readability, as is sometimes done in works of metaphysics (see, e.g., Bacon, 2018; Dorr, 2016; Dorr et al., 2021).

Proof.

(1)	$\phi \approx (\phi \wedge \phi)$	Assumption (to be refuted)
(2)	$\phi \approx \phi$	Ref
(3)	$\phi \prec (\phi \wedge \phi)$	CG
(4)	$(\phi \prec \phi) \approx (\phi \prec (\phi \wedge \phi))$	IdSt_2 1, 2
(5)	$(\phi \prec (\phi \wedge \phi)) \approx (\phi \prec \phi)$	Sym 4
(6)	$\phi \prec \phi$	IdTr 3, 5
(7)	$\phi \nprec \phi$	IG
(8)	$\bot$	PC 6, 7

□

- $(\neg\phi \approx \neg\psi) \to (\phi \approx \psi)$

Proof.

(1)	$\neg\phi \approx \neg\psi$	Assumption
(2)	$\phi \approx \phi$	Ref
(3)	$\neg\neg\phi \approx \neg\neg\psi$	IdSt_1 1
(4)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \neg\neg\psi)$	IdSt_2 2, 3
(5)	$\phi \prec \neg\neg\phi$	NG
(6)	$\phi \prec \neg\neg\psi$	IdTr 4,5
(7)	$\phi \prec \neg\neg\psi \to \phi \approx \psi$	NG
(8)	$\phi \approx \psi$	MP 6, 7

□

- $(\gamma \not\approx \psi) \to (\neg\phi \not\approx (\gamma \wedge \psi))$

Proof.

(1)	$\gamma \not\approx \psi$	Assumption
(2)	$\neg\phi \approx \gamma \wedge \psi$	Assumption (to be refuted)
(3)	$\neg\neg\phi \approx \neg(\gamma \wedge \psi)$	IdSt_1 2
(4)	$\neg(\gamma \wedge \psi) \approx \neg\neg\phi$	Sym 3
(5)	$\neg\gamma \approx \neg\gamma$	Ref
(6)	$\neg\gamma \prec \neg(\gamma \wedge \psi)$	NCG
(7)	$(\neg\gamma \prec \neg\neg\phi) \approx (\neg\gamma \prec \neg(\gamma \wedge \psi))$	IdSt_2 5, 3
(8)	$(\neg\gamma \prec \neg(\gamma \wedge \psi)) \approx (\neg\gamma \prec \neg\neg\phi)$	Sym 7
(9)	$\neg\gamma \prec \neg\neg\phi$	IdTr 6, 8
(10)	$\neg\gamma \approx \phi$	NG 9
(11)	$\neg\psi \prec \neg(\gamma \wedge \psi)$	NCG
(12)	$\neg\psi \approx \neg\psi$	Ref
(13)	$(\neg\psi \prec \neg\neg\phi) \approx (\neg\psi \prec \neg(\gamma \wedge \psi))$	IdSt_2 12, 3
(14)	$(\neg\psi \prec \neg(\gamma \wedge \psi)) \approx (\neg\psi \prec \neg\neg\phi)$	Sym 13
(15)	$(\neg\psi \prec \neg\neg\phi)$	IdTr 11, 14
(16)	$\neg\psi \approx \phi$	NG 15
(17)	$\phi \approx \neg\psi$	Sym 16
(18)	$\neg\gamma \approx \neg\psi$	TR 10, 17
(19)	$\gamma \approx \psi$	Theorem 3.1.3 18
(20)	$\bot$	PC 1, 19

□

- $\neg\phi \not\approx (\phi \land \phi)$

  Proof.

(1)	$\neg\phi \approx \phi \land \phi$	Assumption (to be refuted)
(2)	$\neg\neg\phi \approx \neg(\phi \land \phi)$	IdSt$_1$ 1
(3)	$\phi \approx \phi$	Ref
(4)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \neg(\phi \land \phi))$	IdSt$_2$ 3, 2
(5)	$\phi \prec \neg\neg\phi$	NG
(6)	$\phi \prec \neg(\phi \land \phi)$	IdTr 4, 5
(7)	$\phi \approx \neg\phi$	NCG 6
(8)	$\neg\phi \approx \neg\neg\phi$	IdSt$_1$ 7
(9)	$\phi \approx \neg\neg\phi$	TR 7, 8
(10)	$\phi \prec \neg\neg\phi$	NG
(11)	$(\phi \prec \phi) \approx (\phi \prec \neg\neg\phi)$	IdSt$_2$ 3, 10
(12)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \phi)$	Sym 11
(13)	$\phi \prec \phi$	IdTr 10, 12
(14)	$\phi \nprec \phi$	NG
(15)	$\bot$	PC 13, 14 □

- $\big((\phi \not\approx \psi) \land ((\phi \circ \psi) \approx (\gamma \circ \theta))\big) \to \big((\phi \approx \gamma) \land (\psi \approx \theta)\big) \lor \big((\phi \approx \theta) \land (\psi \approx \gamma)\big)$

  Proof.

(1)	$(\phi \not\approx \psi) \land ((\phi \land \psi) \approx (\gamma \land \theta))$	Assumption
(2)	$\phi \not\approx \psi$	PC 1
(3)	$(\phi \land \psi) \approx (\gamma \land \theta)$	PC 1
(4)	$\neg(\phi \land \psi) \approx \neg(\gamma \land \theta)$	IdSt$_1$ 3
(5)	$\neg\neg(\phi \land \psi) \approx \neg\neg(\gamma \land \theta)$	IdSt$_1$ 4
(6)	$\neg\phi \prec \neg(\phi \land \psi)$	NCG
(7)	$\neg\phi \approx \neg\phi$	Ref
(8)	$(\neg\phi \prec \neg(\phi \land \psi)) \approx (\neg\phi \prec \neg(\gamma \land \theta))$	IdSt$_2$ 7, 4
(9)	$\neg\phi \prec \neg(\gamma \land \theta)$	IdTr 6, 8
(10)	$(\neg\phi \approx \neg\gamma) \lor (\neg\phi \approx \neg\theta)$	NCG 9
(11)	$(\phi \approx \gamma) \lor (\phi \approx \theta)$	PC, Theorem 3.1.3 10
(12)	$\neg\psi \prec \neg(\phi \land \psi)$	NCG
(13)	$\neg\psi \approx \neg\psi$	Ref
(14)	$(\neg\psi \prec \neg(\phi \land \psi)) \approx (\neg\psi \prec \neg(\gamma \land \theta))$	IdSt$_2$ 13, 4
(15)	$\neg\psi \prec \neg(\gamma \land \theta)$	IdTr 12, 14
(16)	$(\neg\psi \approx \neg\gamma) \lor (\neg\psi \approx \neg\theta)$	NCG 15
(17)	$(\psi \approx \gamma) \lor (\psi \approx \theta)$	PC, Theorem 3.1.3 16
(18)	$((\phi \approx \gamma) \lor (\phi \approx \theta)) \land ((\psi \approx \gamma) \lor (\psi \approx \theta))$	PC 11, 17
(19)	$((\phi \approx \theta) \land (\psi \approx \gamma)) \lor ((\phi \approx \gamma) \land (\psi \approx \theta))$	PC 2, 18 □

  Note that, applying basic laws of propositional calculus we find out that (17) is equivalent to $((\phi \approx \gamma) \land (\psi \approx \gamma)) \lor ((\phi \approx \theta) \land (\psi \approx \gamma)) \lor ((\phi \approx \gamma) \land (\psi \approx \theta)) \lor ((\phi \approx \theta) \land (\psi \approx \theta))$, but the first and the fourth disjuncts from this disjunction will not be the case due to (2); that's how we get to (19) by PC.

- $((\phi \wedge \phi) \approx (\gamma \wedge \gamma)) \to (\phi \approx \gamma)$

  Proof.

(1)	$(\phi \wedge \phi) \approx (\gamma \wedge \gamma)$	Assumption
(2)	$\phi \approx \phi$	REF
(3)	$(\phi \prec (\phi \wedge \phi)) \approx (\phi \prec (\gamma \wedge \gamma))$	IDST$_2$ 2, 1
(4)	$(\phi \prec (\phi \wedge \phi)$	CG
(5)	$\phi \prec (\gamma \wedge \gamma)$	IDTR 3, 4
(6)	$(\phi \approx \gamma) \vee (\phi \approx \gamma)$	CG 5
(7)	$\phi \approx \gamma$	PC 6

  □

□

Proof (**Theorem 4.1**). To save space, I mainly focus on the non-factive logic immediate ground (UNIG) and prove the validity of IG, CG, NDG and NG as samples; I also prove TRG$_m$ as a sample for mediate grounding principles. The rest of the principles and logics are done similarly, by direct use of the definitions.

IG. For an arbitrary model $M = (\mathcal{D}_{\langle\rangle}, at, a)$, suppose on the contrary that $M \vDash \phi \prec \phi$. Then $T_{at}([\![\phi \prec \phi]\!]_a) = 1$, hence $[\![\phi]\!]_a \sqsubset [\![\phi]\!]_a$. Let $[\![\phi]\!]_a := A$. According to Definition 4.2 (Grounding Constituency), the following constitute all the possible cases: **(i)** $A = A \circ C$ or $A = C \circ A$ for some $C \in \mathcal{D}_{\langle\rangle}$; **(ii)** $A = \neg A^*$ and either $A = \neg(A^* \circ C)$ or $A = \neg(C \circ A^*)$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$; **(iii)** $A = \neg\neg A$. All these cases are impossible due to the unique readability of the propositions. In other words, as was mentioned before (see footnote 10), since by Definition 4.1, our propositions are as structured as sentences, all these five cases fail due to structural mismatch.

CG ($\Leftarrow$). For an arbitrary model $M = (\mathcal{D}_{\langle\rangle}, at, a)$, suppose $M \vDash \phi \approx \psi \vee \phi \approx \gamma$. Then $M \vDash \phi \approx \psi$ or $M \vDash \phi \approx \gamma$. In the first case we have $T_{at}([\![\phi \approx \psi]\!]_a) := T_{at}([\![\phi]\!]_a \thickapprox [\![\psi]\!]_a) = 1$, so $[\![\phi]\!]_a = [\![\psi]\!]_a$, thus $[\![\phi \wedge \gamma]\!]_a := [\![\phi]\!]_a \wedge [\![\gamma]\!]_a = [\![\psi]\!]_a \wedge [\![\gamma]\!]_a := [\![\psi \wedge \gamma]\!]_a$, and hence $[\![\phi]\!]_a \sqsubset [\![\psi \wedge \gamma]\!]_a$, because $[\![\phi]\!]_a \sqsubset [\![\phi \wedge \psi]\!]_a$. So, we have $T_{at}([\![\phi]\!]_a \prec [\![\psi \wedge \gamma]\!]_a) := T_{at}([\![\phi \prec \psi \wedge \gamma]\!]_a) = 1$. Similarly for the second case. So, in either case we have $M \vDash \phi \prec (\psi \wedge \gamma)$. □

CG ($\Rightarrow$). For an arbitrary model M, suppose $M \vDash \phi \prec (\psi \wedge \gamma)$. Then we have $[\![\phi]\!]_a \sqsubset [\![\psi \wedge \gamma]\!]_a$. By Definition 4.2, one of the following must hold: **(i)** $[\![\psi]\!]_a \wedge [\![\gamma]\!]_a = [\![\phi]\!]_a \circ B$ or $[\![\psi]\!]_a \wedge [\![\gamma]\!]_a = B \circ [\![\phi]\!]_a$, for some $\circ \in \{\wedge, \vee\}$ and $B \in \mathcal{D}_{\langle\rangle}$; **(ii)** $[\![\phi]\!]_a = \neg A^*$ and either $[\![\psi]\!]_a \wedge [\![\gamma]\!]_a = \neg(A^* \circ B^*)$ or $[\![\psi]\!]_a \wedge [\![\gamma]\!]_a = \neg(B^* \circ A^*)$, for some $\circ \in \{\wedge, \vee\}$ and $A^*, B^* \in \mathcal{D}_{\langle\rangle}$; **(iii)** $[\![\psi]\!]_a \wedge [\![\gamma]\!]_a = \neg\neg[\![\phi]\!]_a$. All options packed in **(ii)** and **(iii)** are immediately ruled out by the corresponding propositional non-identities due to structural differences. That leaves us with **(i)**. Again, the only possible identity for **(i)** is when $\circ = \wedge$. It follows that either $[\![\psi]\!]_a = [\![\phi]\!]_a$ or $[\![\gamma]\!]_a = [\![\phi]\!]_a$. Hence $M \vDash \psi \approx \phi \vee \gamma \approx \phi$.

NDG (⇒). For an arbitrary model M, suppose $M \vDash \phi \prec \neg(\psi \vee \gamma)$. Then we have $[\![\phi]\!]_a \sqsubset \neg([\![\psi]\!]_a \vee [\![\gamma]\!]_a)$. From all the possible cases of constituency, only the following are structurally possible:

(iii) $[\![\phi]\!]_a = A^*$ and $\neg([\![\psi]\!]_a \vee [\![\gamma]\!]_a) = \neg(A^* \vee C)$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$,

(iv) $[\![\phi]\!]_a = A^*$ and $\neg([\![\psi]\!]_a \vee [\![\gamma]\!]_a) = \neg(C \vee A^*)$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$.

So either of these two can hold. Now, from the first one it follows that $[\![\psi]\!]_a = A^*$, so $[\![\neg\psi]\!]_a := \neg[\![\psi]\!]_a = \neg A^* = [\![\phi]\!]_a$, and thus $M \vDash (\psi \approx \neg\phi)$. In a similar manner, from the second one it follows that $M \vDash (\gamma \approx \neg\phi)$. As a result, it follows that $M \vDash (\psi \approx \neg\phi) \vee (\gamma \approx \neg\phi)$.

NDG (⇐). This side holds because for any given assignment function a, we have $[\![\phi]\!]_a \sqsubset [\![\neg(\phi \vee \gamma)]\!]_a$ and $[\![\gamma]\!]_a \sqsubset [\![\neg(\phi \vee \gamma)]\!]_a$.

NG (⇒). For an arbitrary model M, suppose $M \vDash \psi \prec \neg\neg\phi$. The only structurally possible case is that $\neg\neg[\![\psi]\!]_a = \neg\neg[\![\phi]\!]_a$, thus $[\![\psi]\!]_a = [\![\phi]\!]_a$, and hence $M \vDash (\psi \approx \phi)$.

NG (⇐). This side holds, because $[\![\phi]\!]_a \sqsubset \neg\neg[\![\phi]\!]_a$ for any assignment a.

TRG$_m$. The aim is to show $M \vDash ((\phi \prec_m \psi) \wedge (\psi \prec_m \theta)) \rightarrow \phi \prec_m \theta$ for an arbitrary model $M = (\mathcal{D}_{\langle\rangle}, at, a)$. That is, we need to show that if $M \vDash ((\phi \prec_m \psi) \wedge (\psi \prec_m \theta))$, then $M \vDash \phi \prec_m \theta$. Suppose $M \vDash ((\phi \prec_m \psi) \wedge (\psi \prec_m \theta))$. It follows that $M \vDash \phi \prec_m \psi$ and $M \vDash \psi \prec_m \theta$. From the first relation it follows that $[\![\phi]\!]_a \sqsubset^* [\![\psi]\!]_a$, where $\sqsubset^*$ is the transitive closure of the grounding constituency relation $\sqsubset$; from the second it follows that $[\![\psi]\!]_a \sqsubset^* [\![\theta]\!]_a$. Since $\sqsubset^*$ is a transitive closure, it follows that $[\![\phi]\!]_a \sqsubset^* [\![\theta]\!]_a$ (see Footnote 14), which means that $M \vDash \phi \prec_m \theta$.

□

To prove Theorem 4.2, we need the following lemma:

Lemma. *If $a^* : AT \rightarrow \mathcal{D}_{\langle\rangle}$ is the assignment function such that $a^*(p_i) = P_i$, the induced interpretation function $[\![.]\!]_{a^*} : \mathcal{L} \rightarrow \mathcal{D}_{\langle\rangle}$ is one to one.*

Proof. Induction on the structure of propositions in $\mathcal{D}_{\langle\rangle}$ (surjection) and on the structure of formulas in $\mathcal{L}$ (injection). □

We call the assignment function described in the lemma *straightforward*.

Proof (**Theorem 4.2**). **(i)** Suppose $\vDash \phi \prec \psi$. Then $[\![\phi]\!]_a \sqsubset [\![\psi]\!]_a$ for every assignment function a. Consider, in particular, the straightforward assignment function a^*. By Definition 4.2 there are 5 possible general cases. We only prove the claim for one of them; the rest are proved similarly, using the fact that a^* is 1-1 function.

$[\![\psi]\!]_{a^*} = [\![\phi]\!]_{a^*} \circ B$ or $[\![\psi]\!]_{a^*} = B \circ [\![\phi]\!]_{a^*}$ for some $B \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$. Consider the first case. By Lemma, $B = [\![\gamma]\!]_{a^*}$ for some $\gamma \in \mathcal{L}$. So we have $[\![\phi]\!]_{a^*} \circ B = [\![\phi]\!]_{a^*} \circ [\![\gamma]\!]_{a^*}$, and hence $[\![\psi]\!]_{a^*} = [\![\phi]\!]_{a^*} \circ [\![\gamma]\!]_{a^*} = [\![\phi \circ \gamma]\!]_{a^*}$. Since a injective (Lemma), we have it that ψ is syntactically identical to $\phi \circ \gamma$, hence by Ref, we have $\vdash \psi \approx \phi \circ \gamma$. Now, since by CG and DG (depending on the choice of $\circ$) we have $\vdash \phi \prec (\phi \circ \gamma)$, it follows by Str_1 that $\vdash \phi \prec (\phi \circ \gamma) \approx \phi \prec \psi$. From propositional calculus (PC) and Str_1 it follows that $\vdash \phi \prec \psi$. □

(iii) Suppose $\vDash \phi \prec_m \psi$. Then $[\![\phi]\!]_a \sqsubset^* [\![\psi]\!]_a$ for every assignment function a. So for every assignment a there are propositions $A_1, A_2, ...A_n \in \mathcal{D}_{\langle\rangle}$ such that $[\![\phi]\!]_a \sqsubset A_1 \sqsubset A_2 \sqsubset ... \sqsubset A_n \sqsubset [\![\psi]\!]_a$. Now, consider the straightforward assignment function a^*. Due to Lemma, for each $i = 1, ..., n$, we have $A_i = [\![\gamma_i]\!]_{a^*}$ for some formula γ_i. Thus our 'chain' of immediate grounding relations turns into $[\![\phi]\!]_{a^*} \sqsubset [\![\gamma_1]\!]_{a^*} \sqsubset [\![\gamma_2]\!]_{a^*} \sqsubset ... \sqsubset [\![\gamma_n]\!]_{a^*} \sqsubset [\![\psi]\!]_{a^*}$. From the proof of case (i) above, we have $\vdash \phi \prec \gamma_1$, $\vdash \gamma_1 \prec \gamma_2$, ..., $\vdash \gamma_n \prec \psi$. By multiple applications of the transitivity schema (TRG_m) and *modus ponens* (MP), we obtain $\vdash \phi \prec \psi$.

□

Proof (**Theorem 5.1**). Straightforward. I only prove the left-to-right of CoG. For an arbitrary (extended) model M, suppose that $M \vDash \gamma \prec (\phi \rightarrow \psi)$. Then we have $[\![\gamma]\!]_a \sqsubset [\![\phi \rightarrow \psi]\!]_a := [\![\phi]\!]_a \rightarrow [\![\psi]\!]_a$. Given the extended definition of grounding constituency and the sentence-like structure of propositions, the only possible cases here are the following: **(i)** $[\![\gamma]\!]_a = \neg A^*$ and $[\![\phi]\!]_a \rightarrow [\![\psi]\!]_a = A^* \rightarrow C$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$, which results in $[\![\gamma]\!]_a = \neg [\![\phi]\!]_a$, or **(ii)** $[\![\phi]\!]_a \rightarrow [\![\psi]\!]_a = C \rightarrow [\![\gamma]\!]_a$ for some $C \in \mathcal{D}_{\langle\rangle}$, which entails $[\![\gamma]\!]_a = [\![\psi]\!]_a$. Thus either of the identities $[\![\gamma]\!]_a = [\![\neg\phi]\!]_a$ and $[\![\gamma]\!]_a = [\![\psi]\!]_a$ holds, hence $M \vDash \gamma \approx \neg\phi \vee \gamma \approx \psi$. □

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